

On The Best Approximate Solutions of The Matrix Equation $AXB = C$

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Abstract

Suppose that the matrix equation $AXB = C$ with unknown matrix X is given, where A , B , and C are known matrices of suitable sizes. The matrix nearness problem is considered over the general and least squares solutions of the matrix equation $AXB = C$ when the equation is consistent and inconsistent, respectively. The implicit form of the best approximate solutions of the problems over the set of symmetric and the set of skew-symmetric matrices are established as well. Moreover, some numerical examples are given for the problems considered.

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1 Introduction and Notations

Let $\mathcal{R}^{m \times n}$, $\mathcal{SR}^{n \times n}$, and $\mathcal{SSR}^{n \times n}$ be the set of $m \times n$ real matrices, the set of $n \times n$ real symmetric matrices, and the set of $n \times n$ real skew-symmetric matrices, respectively. The symbols A^T , A^\dagger , and $\|A\|$ will denote the transpose, the Moore–Penrose generalized inverse, and the Frobenius norm (see, for example, [7]), respectively, of a matrix $A \in \mathcal{R}^{m \times n}$. Further, $\text{vec}(\cdot)$ will stand for the vec operator, i.e. $\text{vec}(A) = (a_1^T, a_2^T, \dots, a_n^T)^T$ for the matrix $A = (a_1, a_2, \dots, a_n) \in \mathcal{R}^{m \times n}$, $a_i \in \mathcal{R}^{m \times 1}$, $i = 1, 2, \dots, n$, and $A \otimes B$ will stand for the Kronecker product of matrices A and $B \in \mathcal{R}^{m \times n}$, (see [5]).

The well-known linear matrix equation $AXB = C$, where A , B , C are known matrices of suitable sizes and X is the matrix of unknowns, were studied in the case of special solution structures, e.g. symmetric, triangular or diagonal solution X in [1, 2, 3, 4, 15, 16] using matrix decomposition such as the singular value decomposition (SVD), the generalized SVD, the quotient SVD, and the canonical correlation decomposition. In these literatures, the matrix equation $AXB = C$ is consistent. But, it is rarely possible to satisfy the consistency condition of the matrix equation $AXB = C$, since the matrices A , B , and C occurring in practice are usually obtained from an experiment.

An iteration method to solve the linear matrix equation $AXB = C$ over the set of symmetric matrices have constructed by Peng et al. [17]. In addition,

Peng [19] has established an iterative method to solve the minimum Frobenius norm residual problem: $\min \|AXB - C\|$ where X is the symmetric matrix of unknowns. Huang and Yin solved the constrained inverse eigenproblem and associated approximation problem for anti-Hermitian R -symmetric matrices and the matrix inverse problem and its optimal approximation problem for R -symmetric matrices in [8] and [9], respectively. Huang et al. gave the precise solutions to the minimum residual problem and the matrix nearness problem for symmetric matrices or skew-symmetric matrices in [10] and constructed an iterative method to solve the linear matrix equation $AXB = C$ over the set of skew-symmetric matrices in [11].

This work is devoted to give the best approximate solutions of the following two problems, which are interesting and known as the matrix nearness problems, in an alternative way:

Problem 1 For given matrices $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{p \times r}$, and $C \in \mathcal{R}^{m \times r}$, let S_G be the set of all solutions of the consistent matrix equation

$$AXB = C.$$

For a given matrix $X_0 \in \mathcal{R}^{n \times p}$, find $\hat{X} \in S_G$ such that

$$\|\hat{X} - X_0\| = \min_{X \in S_G} \|X - X_0\|.$$

Problem 2 For given matrices $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{p \times r}$, and $C \in \mathcal{R}^{m \times r}$, let S_E be the set of all least squares solutions of the minimum residual problem

$$\min_{X \in \mathcal{R}^{n \times p}} \|AXB - C\|.$$

For a given matrix $X_0 \in \mathcal{R}^{n \times p}$, find $\hat{X} \in S_E$ such that

$$\|\hat{X} - X_0\| = \min_{X \in S_E} \|X - X_0\|.$$

In fact, the Problems 1 and 2 are to find the best approximate solution for a given matrix $X_0 \in \mathcal{R}^{n \times p}$ over the set of general solutions (S_G) and the least square solutions (S_E) of the matrix equation $AXB = C$, respectively. These problems are known as the matrix nearness problem in the literature. The matrix nearness problem is very important in applied sciences and has been extensively studied in recent years (see, for example, [6, 12, 13, 18]). Therefore, it is important to give the best approximate solutions of the problems in implicit forms.

In general, numerical algorithms or iteration methods for solving these problems are suggested in most of the works mentioned above.

In this work, the implicit forms of the best approximate solutions to the problems mentioned above have been obtained over the set of symmetric and the set of skew-symmetric matrices using the Moore–Penrose generalized inverse. Moreover, some numerical examples are given via Matlab 7.5. The matrices in the examples have been taken from related reference work articles.

2 Preliminary Results

The vector $x_0 \in \mathcal{R}^{n \times 1}$ is the best approximate solution (BAS) to the inconsistent system of linear equations $Ax = g$, where $A \in \mathcal{R}^{m \times n}$, if and only if

- 1- $(Ax - g)^T (Ax - g) \geq (Ax_0 - g)^T (Ax_0 - g)$ for all $x \in \mathcal{R}^{n \times 1}$,
- 2- $x^T x > x_0^T x_0$ for all $x \in \mathcal{R}^{n \times 1} \setminus \{x_0\}$ satisfying $(Ax - g)^T (Ax - g) = (Ax_0 - g)^T (Ax_0 - g)$ [5].

The vector $x_0 \in \mathcal{R}^{n \times 1}$ is a least squares solution (LSS) to the inconsistent system of linear equations $Ax = g$, where $A \in \mathcal{R}^{m \times n}$, if and only if

$$(Ax - g)^T (Ax - g) \geq (Ax_0 - g)^T (Ax_0 - g)$$

for all $x \in \mathcal{R}^{n \times 1}$ [5].

It is noteworthy that there may be many LSS for an inconsistent system of linear equations. In addition, an LSS may not be the BAS while the BAS is always a LSS. However, the BAS is always unique.

We close this section by giving two auxiliary results related to the problems mentioned earlier and which will be used in the rest of the work.

Lemma 1 *Suppose that $S_{G'}$ is the set of all solutions to the consistent system of linear equations $Ax = g$, where $A \in \mathcal{R}^{m \times n}$ is a known matrix, $g \in \mathcal{R}^{m \times 1}$ is a known vector, and $x \in \mathcal{R}^{n \times 1}$ is the vector of unknowns. For a given vector $x_0 \in \mathcal{R}^{n \times 1}$, the vector $\hat{x} \in S'_{G'}$ satisfying*

$$\|\hat{x} - x_0\| = \min_{x \in S_{G'}} \|x - x_0\|$$

is given by

$$\hat{x} = A^\dagger g + (I - A^\dagger A) x_0.$$

Proof. If $x \in S_{G'}$, then it can be written in the form

$$x = A^\dagger g + (I - A^\dagger A) h$$

for some vector $h \in \mathcal{R}^{n \times 1}$ [5, Theorem 6.3.2]. Thus, the problem turns into the problem of finding the BAS \hat{x} of the system

$$A^\dagger g + (I - A^\dagger A) h = x_0 \tag{2.1}$$

or equivalently the system

$$(I - A^\dagger A) h = x_0 - A^\dagger g.$$

Since the matrix $I - A^\dagger A$ is symmetric and idempotent, it is obtained

$$\hat{h} = (I - A^\dagger A) x_0$$

by Theorem 7.4.1 in [5]. Substituting this expression in the equation (2.1), we get

$$\hat{x} = A^\dagger g + (I - A^\dagger A) x_0.$$

So, the proof is completed. ■

Lemma 2 Let $S_{E'}$ be the set of all least squares solutions to the system of linear equations $Ax = g$ which do not need to be consistent, where $A \in \mathcal{R}^{m \times n}$ is a known matrix, $g \in \mathcal{R}^{m \times 1}$ is a known vector, and $x \in \mathcal{R}^{n \times 1}$ is the vectors of unknowns. For a given vector $x_0 \in \mathcal{R}^{n \times 1}$, the vector $\hat{x} \in S_{E'}$ satisfying

$$\|\hat{x} - x_0\| = \min_{x \in S_{E'}} \|x - x_0\|$$

is given by

$$\hat{x} = A^\dagger g + (I - A^\dagger A) x_0.$$

Proof. If $x \in S_{E'}$, then it can be written in the form

$$x = A^\dagger g + (I - A^\dagger A) h$$

for some vector $h \in \mathcal{R}^{n \times 1}$ [5, Theorem 6.3.2]. This is of the same type with (2.1). So, the remaining part of the proof can be completed easily in a similar way as in the proof of Lemma 1. ■

It is noteworthy that the structures of \hat{x} in Lemmas 1 and 2 are exactly the same.

3 The Best Approximate Solutions of Problems 1 and 2

If it is assumed that the matrix equation $AXB = C$, where $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{p \times r}$, $C \in \mathcal{R}^{m \times r}$ are known nonzero matrices and $X \in \mathcal{R}^{n \times p}$ is the matrix of unknowns, is inconsistent, as was the system of linear equations, then it may be asked to find a matrix X such that $\|AXB - C\|$ is minimum, too. A matrix satisfying this condition is called an approximate solution to the matrix equation. The matrix $\hat{X} \in \mathcal{R}^{n \times p}$ is defined to be the BAS to the matrix equation $AXB = C$ if and only if

- 1- $\|AXB - C\| \geq \|A\hat{X}B - C\|$ for all $X \in \mathcal{R}^{n \times p}$;
- 2- $\|X\| > \|\hat{X}\|$ for all $X \in \mathcal{R}^{n \times p} \setminus \{\hat{X}\}$ satisfying $\|AXB - C\| = \|A\hat{X}B - C\|$.

We note that a vector $k \in \mathcal{R}^{mn \times 1}$ will stand for the vector $\text{vec}(K)$ in the rest of the text, where $K \in \mathcal{R}^{m \times n}$.

It is known that the matrix equation $AXB = C$ can be equivalently written as

$$(B^T \otimes A) x = c, \tag{3.1}$$

where $(B^T \otimes A)$ is the kronecker product (see, for detail, [20]). Consequently, the solutions of a matrix equation $AXB = C$ can be obtained considering the usual system of linear equations (3.1) instead of the matrix equation $AXB = C$.

Now we can give the solutions of Problems 1 and 2 which are the subjects of the following two theorems, respectively.

Theorem 1 Let the matrix equation $AXB = C$ be consistent, where $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{p \times r}$, $C \in \mathcal{R}^{m \times r}$ are known nonzero matrices and $X \in \mathcal{R}^{n \times p}$ is the matrix of unknowns. Then, for a given matrix $X_0 \in \mathcal{R}^{n \times p}$, the matrix $\hat{X} \in S_G$ satisfying

$$\|\hat{X} - X_0\| = \min_{X \in S_G} \|X - X_0\|$$

is given by

$$\hat{X} = A^\dagger C B^\dagger + X_0 - A^\dagger A X_0 B B^\dagger.$$

Proof. If $X \in S_G$, then it can be written in the form

$$X = A^\dagger C B^\dagger + H - A^\dagger A H B B^\dagger \quad (3.2)$$

for some matrix $H \in \mathcal{R}^{n \times p}$ [20]. Since the statement (3.2) is equivalent to the statement

$$x = (B^T \otimes A)^\dagger c + \left[I - (B^T \otimes A)^\dagger (B^T \otimes A) \right] h,$$

the problem turns into the problem of finding the best approximate solution of the usual system of linear equations

$$(B^T \otimes A)^\dagger c + \left[I - (B^T \otimes A)^\dagger (B^T \otimes A) \right] h = x_0$$

or equivalently

$$\left[I - (B^T \otimes A)^\dagger (B^T \otimes A) \right] h = x_0 - (B^T \otimes A)^\dagger c. \quad (3.3)$$

Using Lemma 1 and (3.3), we get

$$\hat{x} = (B^T \otimes A)^\dagger c + \left[I - (B^T \otimes A)^\dagger (B^T \otimes A) \right] x_0$$

or, in the matrix form,

$$\hat{X} = A^\dagger C B^\dagger + X_0 - A^\dagger A X_0 B B^\dagger.$$

So, the proof is completed. ■

Lemma 3 Let the matrices A , B , and C be as in Theorem 1, and assume that the matrix X is a least squares solution to the inconsistent matrix equation $AXB = C$. Then the matrix X can be written in the form

$$X = A^\dagger C B^\dagger + H - A^\dagger A H B B^\dagger$$

for some matrix $H \in \mathcal{R}^{n \times p}$.

Proof. The proof immediately follows from Theorem 6.3.2 and Theorem 7.6.3 in [5] considering the usual system of linear equations $(B^T \otimes A)x = c$ instead of the matrix equations $AXB = C$ because the former and the latter are equivalent. ■

Theorem 2 Let the matrix equation $AXB = C$ be inconsistent, where $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{p \times r}$, $C \in \mathcal{R}^{m \times r}$ are known nonzero matrices and $X \in \mathcal{R}^{n \times p}$ is the matrix of unknowns. Then, for a given matrix $X_0 \in \mathcal{R}^{n \times p}$, the matrix $\hat{X} \in S_E$ satisfying

$$\|\hat{X} - X_0\| = \min_{X \in S_E} \|X - X_0\|$$

is given by

$$\hat{X} = A^\dagger C B^\dagger + X_0 - A^\dagger A X_0 B B^\dagger.$$

Proof. By Lemma 3, any least squares solutions of the inconsistent matrix equation $AXB = C$ is in the form

$$X = A^\dagger CB^\dagger + H - A^\dagger AHBB^\dagger$$

for some matrix $H \in \mathcal{R}^{n \times p}$. Hence, in the framework of Lemma 2, the proof is easily completed by proceeding as in the proof of Theorem 1. ■

4 The Symmetric and Skew-Symmetric Solutions of Problems 1 and 2

Now, suppose that the symmetric solutions of Problems 1 and 2 are required. To do this, the pair of matrix equations

$$\begin{aligned} AXB &= C \\ B^T X A^T &= C^T \end{aligned}$$

or, equivalently, the usual system of linear equations

$$\begin{bmatrix} B^T \otimes A \\ A \otimes B^T \end{bmatrix} x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

instead of the matrix equation $AXB = C$ is taken with $c_1 = \text{vec}(C)$ and $c_2 = \text{vec}(C^T)$. Then, for a given matrix $X_0 \in S\mathcal{R}^{n \times n}$, in the framework of Theorems 1 and 2 the solution matrix $\hat{X} \in S\mathcal{R}^{n \times n}$ is obtained by using $\hat{x} = \text{vec}(\hat{X})$ given by

$$\hat{x} = \begin{bmatrix} B^T \otimes A \\ A \otimes B^T \end{bmatrix}^\dagger \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + x_0 - \begin{bmatrix} B^T \otimes A \\ A \otimes B^T \end{bmatrix}^\dagger \begin{bmatrix} B^T \otimes A \\ A \otimes B^T \end{bmatrix} x_0. \quad (4.1)$$

Remark 1 If the matrix X_0 is not symmetric, then the matrix $\frac{1}{2}(X_0 + X_0^T)$, instead of the matrix X_0 , is taken to find the symmetric solutions of Problems 1 or 2. The reason for this is that the minimization problem

$$\min \|X - X_0\|$$

is equivalent the minimization problem

$$\min \left\| X - \frac{1}{2}(X_0 + X_0^T) \right\|$$

over the subset S_G or S_E in $S\mathcal{R}^{n \times n}$ since

$$\|X - X_0\|^2 = \left\| X - \frac{1}{2}(X_0 + X_0^T) \right\|^2 + \left\| \frac{1}{2}(X_0 - X_0^T) \right\|^2, \forall X \in S\mathcal{R}^{n \times n}.$$

(for example, see [14, 17, 19] for details).

Now suppose that the skew-symmetric solutions of Problems 1 and 2 are required. To do this, the pair of matrix equations

$$\begin{aligned} AXB &= C \\ B^T X A^T &= -C^T \end{aligned}$$

or, equivalently, the usual system of linear equations

$$\begin{bmatrix} B^T \otimes A \\ A \otimes B^T \end{bmatrix} x = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix}$$

instead of the matrix equation $AXB = C$ is taken where c_1 and c_2 are as in Remark 1. Then, for a given matrix $X_0 \in SSR^{n \times n}$, the solution matrix $\hat{X} \in SSR^{n \times n}$ is obtained by using $\hat{x} = \text{vec}(\hat{X})$ given by

$$\hat{x} = \begin{bmatrix} B^T \otimes A \\ A \otimes B^T \end{bmatrix}^\dagger \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix} + x_0 - \begin{bmatrix} B^T \otimes A \\ A \otimes B^T \end{bmatrix}^\dagger \begin{bmatrix} B^T \otimes A \\ A \otimes B^T \end{bmatrix} x_0. \quad (4.2)$$

Remark 2 *If the matrix X_0 is not skew-symmetric, then the matrix $\frac{1}{2}(X_0 - X_0^T)$ instead of the matrix X_0 is taken to find the skew-symmetric solutions of the Problems 1 or 2. The reason for this is that the minimization problem*

$$\min \|X - X_0\|$$

is equivalent to the minimization problem

$$\min \left\| X - \frac{1}{2}(X_0 - X_0^T) \right\|$$

over the subset S_G or S_E in $SSR^{n \times n}$ since

$$\|X - X_0\|^2 = \left\| X - \frac{1}{2}(X_0 - X_0^T) \right\|^2 + \left\| \frac{1}{2}(X_0 + X_0^T) \right\|^2, \forall X \in SSR^{n \times n},$$

(for example, see [11] for details).

Note that the structures of the general solution and a least squares solution of the matrix equation $AXB = C$ when the equation is consistent and inconsistent, respectively, are exactly the same. Therefore, the structures of the solutions of Problems 1 and 2 are the same, too. The former and the latter facts are immediately seen from Theorem 1 and 2, respectively, together with Lemma 3 in the framework of Lemmas 1 and 2.

We conclude the paper by giving a few numerical examples. As it was mentioned earlier, the matrices in the examples have been taken from related reference work articles. It is seen that the solutions are exactly the same with four decimal digits as those in the works cited. All the computations have been performed using Matlab 7.5.

Example 1 [11, Example 4]. *Consider the skew-symmetric solution of Problem 1, where*

$$A = \begin{bmatrix} 1 & 3 & -5 & 7 & -9 \\ 2 & 0 & 4 & 6 & -1 \\ 0 & -2 & 9 & 6 & -8 \\ 3 & 6 & 2 & 27 & -13 \\ -5 & 5 & -22 & -1 & -11 \\ 8 & 4 & -6 & -9 & -19 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 8 & -5 & 4 \\ -1 & 5 & 0 & -2 & 3 \\ 4 & -1 & 0 & 2 & 5 \\ 0 & 3 & 9 & 2 & -6 \\ -2 & 7 & -8 & 1 & 11 \end{bmatrix},$$

$$C = \begin{bmatrix} 171 & -537 & 74 & -29 & -281 \\ 142 & -278 & 212 & -92 & -150 \\ 196 & -523 & -59 & -111 & 24 \\ 661 & -1507 & 922 & -234 & -1003 \\ -39 & -192 & -207 & 186 & -227 \\ -165 & -292 & -1154 & 76 & 422 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 1 & 0 & 4 & -1 & 0 \\ 5 & 3 & 2 & 7 & 4 \\ -1 & -2 & 0 & -1 & 0 \\ 2 & 6 & 1 & 8 & -4 \\ 0 & 3 & 1 & 4 & 2 \end{bmatrix}.$$

By the formula (4.2) in the framework Remark 2, the skew-symmetric solution matrix is obtained as

$$\hat{X} = \begin{bmatrix} 0.0000 & 2.0000 & -1.0000 & -2.0000 & 0.0000 \\ -2.0000 & 0.0000 & 2.0000 & 1.0000 & -4.0000 \\ 1.0000 & -2.0000 & 0.0000 & -1.0000 & -0.0000 \\ 2.0000 & -1.0000 & 1.0000 & 0.0000 & -4.0000 \\ 0.0000 & 4.0000 & -0.0000 & 4.0000 & 0.0000 \end{bmatrix}.$$

Example 2 [14, Example 1]. Consider the symmetric solution of Problem 2 where

$$A = \begin{pmatrix} E_{55} & Z_{54} \\ Z_{45} & P_4 \end{pmatrix}, \quad B = \begin{pmatrix} K_4 & Z_{45} \\ Z_{54} & Z_{55} \end{pmatrix},$$

$$C = \begin{pmatrix} T_4 & Z_{45} \\ Z_{54} & H_5 \end{pmatrix}, \quad X_0 = \begin{pmatrix} I_4 & \frac{1}{2}E_{45} \\ \frac{1}{2}E_{54} & I_5 \end{pmatrix}.$$

Here E_{mn} and Z_{mn} are $m \times n$ matrices whose all entries 1 and 0, respectively, and P_4 and H_5 denote the 4×4 symmetric Pascal matrix and the 5×5 Hilbert matrix, respectively. Moreover, T_4 and K_4 are Toeplitz and Hankel matrices given by

$$T_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad K_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix},$$

respectively. By the formula (4.1) the symmetric solution matrix is obtained as

$$\hat{X} = \begin{bmatrix} 0.8258 & -0.2692 & -0.2480 & -0.2214 & 0.4129 & 0 & 0 & 0 & 0 \\ -0.2692 & 0.6358 & -0.3430 & -0.3164 & 0.3179 & 0 & 0 & 0 & 0 \\ -0.2480 & -0.3430 & 0.6783 & -0.2952 & 0.3391 & 0 & 0 & 0 & 0 \\ -0.2214 & -0.3164 & -0.2952 & 0.7314 & 0.3657 & 0 & 0 & 0 & 0 \\ 0.4129 & 0.3179 & 0.3391 & 0.3657 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 3 [19, Example 3.1]. Consider the symmetric solution of Problem 2,

where

$$A = \begin{bmatrix} 4 & 3 & -1 & 3 & 1 & -3 & 2 \\ 3 & -2 & 3 & -4 & 3 & 2 & 1 \\ 4 & 3 & -1 & 3 & 1 & -3 & 2 \\ 3 & -1 & 3 & -1 & 3 & 2 & 1 \\ 4 & 3 & -1 & 3 & 1 & -3 & 2 \\ 3 & -1 & 3 & -1 & 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 4 & -3 & -3 & 4 & 4 \\ 5 & -3 & 5 & 5 & -3 & -3 \\ -6 & 2 & -6 & -6 & 2 & 2 \\ -8 & 4 & -8 & -8 & 4 & 4 \\ 4 & -5 & 4 & 3 & -2 & -7 \\ -3 & 2 & -3 & -3 & 2 & 2 \\ -1 & -2 & -1 & -1 & -2 & -2 \end{bmatrix},$$

$$C = \begin{bmatrix} 43 & -54 & 73 & -54 & 51 & -54 \\ -31 & 37 & -61 & 37 & -53 & 37 \\ 43 & -54 & 73 & -54 & 51 & -54 \\ -31 & 37 & -61 & 37 & -53 & 37 \\ 47 & -54 & 73 & -54 & 21 & -54 \\ -31 & 27 & -61 & 27 & -53 & 27 \end{bmatrix}, \quad X_0 = \begin{bmatrix} -1 & 2 & -3 & 2 & -1 & 1 & 3 \\ 2 & -1 & 3 & -3 & 2 & -3 & 4 \\ -3 & 3 & -3 & 3 & -2 & 1 & -1 \\ 2 & -3 & 3 & 2 & 2 & 2 & 4 \\ -3 & 2 & -2 & 2 & -1 & 3 & -3 \\ 4 & 3 & 1 & 1 & 2 & 1 & 1 \\ 1 & -2 & 1 & 3 & 4 & -1 & 1 \end{bmatrix}.$$

By the formula (4.1) in the framework Remark 1, the symmetric solution matrix is obtained as

$$\hat{X} = \begin{bmatrix} 1.7699 & 1.8581 & -3.5455 & 2.8924 & 0.5920 & 0.8523 & 2.3693 \\ 1.8581 & -0.6722 & 1.8908 & -1.9156 & 3.1173 & 0.5698 & -1.0561 \\ -3.5455 & 1.8908 & -0.5812 & 1.3562 & -3.9861 & 1.7472 & 2.2490 \\ 2.8924 & -1.9156 & 1.3562 & -3.8543 & -0.6224 & 0.3241 & 3.6833 \\ 0.5920 & 3.1173 & -3.9861 & -0.6224 & -2.3618 & -2.2044 & 3.2716 \\ 0.8523 & 0.5698 & 1.7472 & 0.3241 & -2.2044 & -0.0556 & 2.7992 \\ 2.3693 & -1.0561 & 2.2490 & 3.6833 & 3.2716 & 2.7992 & 0.0308 \end{bmatrix}.$$

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